An overview of data-driven model reference control techniques

Grand séminaire MACS

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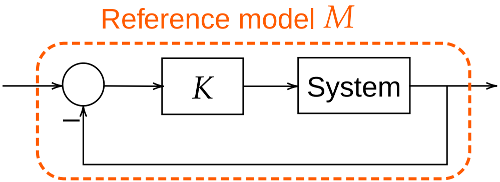
Reference model based data-driven control

Given data from the system P, design K such that the resulting closed-loop is as close as possible to the reference model M



Main techniques:

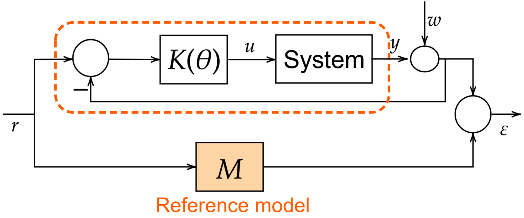
- Iterative Feedback Tuning (IFT)
- Correlation-based Tuning (CbT)
- Virtual Reference Feedback Tuning (VRFT)
- Loewner Data-Driven Control (L-DDC)

Overview of these methods - CbT and IFT

$$\min_{\theta} \left\| \frac{PK(\theta)}{1+PK(\theta)} - M \right\|_2^2$$

$$K(\theta, s) = \frac{b_0 + b_1 s + \cdots + b_n s^n}{1 + a_1 s + \cdots + a_n s^n}$$

$$u(t) = \Phi(t)^T \theta$$



link with closed-loop identification

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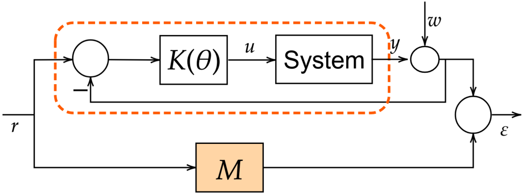

CbT: $\varepsilon(s) = \left(M - \frac{PK(\theta)}{1+PK(\theta)}\right) r(s) + \frac{1}{1+PK(\theta)} w(s)$

- $\min_{\theta} R_{r\varepsilon}(\theta)$

link with closed-loop identification

Iterative correlation-based controller tuning, Karimi et al. (2004).

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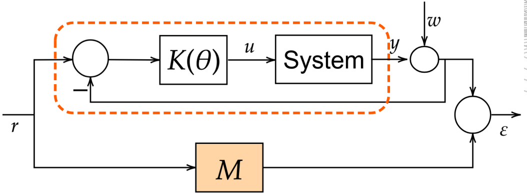

Reference model

CbT: $\varepsilon(s) = \left(M - \frac{PK(\theta)}{1+PK(\theta)}\right) r(s) + \frac{1}{1+PK(\theta)} w(s)$

- $\min_{\theta} R_{r\varepsilon}(\theta)$

IFT: $\frac{\partial \varepsilon}{\partial \theta} = \frac{1}{K(\theta)} \frac{\partial K}{\partial \theta} \left(\frac{PK(\theta)}{1+PK(\theta)} (r - y(\theta)) \right)$

link with closed-loop identification

- $\min_{\theta} R_{r\varepsilon}(\theta)$

Iterative correlation-based controller tuning, Karimi et al. (2004).
Iterative feedback tuning: theory and applications, Hjalmarsson et al. (1998).

Overview of these methods - VRFT

VRFT

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Virtual reference feedback tuning: a direct method for the design of feedback controllers, Campi et al. (2002).

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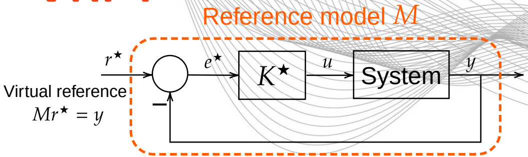

From a single trajectory $(u_k, y_k)_{k=1\ldots N}$:

1. Compute virtual reference $r^\star$: $Mr^\star = y$

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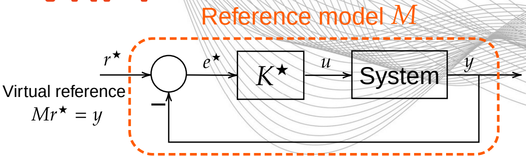

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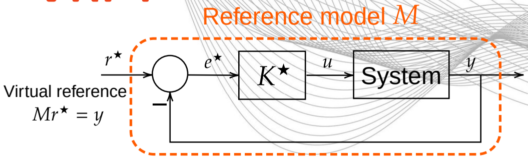

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1. Compute virtual reference $r^\star$: $Mr^\star = y$
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3. Filter to get $e_L^\star$ and u_L

$$\frac{PK(\theta)}{1 + PK(\theta)} - M = \frac{PK(\theta)}{1 + PK(\theta)} - \frac{PK^\star}{1 + PK^\star} \approx \frac{P(K(\theta) - K^\star)}{1 + PK^\star} = P(K(\theta) - K^\star)(1 - M)$$

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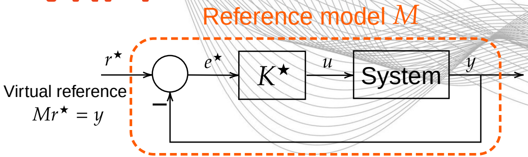

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$$\min_{\theta} J_N^{VRFT}(\theta)$$

$$J_N^{VRFT}(\theta) = \|u_L - K(\theta) e_L^\star\|_2^2$$

$$J_N^{VRFT}(\theta) \underset{N\to\infty}{\rightarrow} J^{VRFT}(\theta) = J(\theta)$$

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Laplace

Overview of these methods - L-DDC

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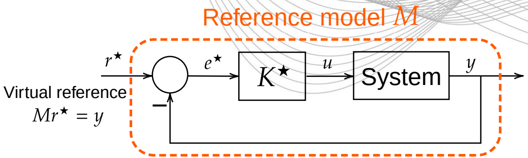
"Free" controller structure (LTI)



L-DDC: $\frac{PK^\star}{1+PK^\star} = M \;\rightarrow\; K(\imath\omega_i) = K^\star(\imath\omega_i)$

Other sysid/approximation techniques can be used (subspace for instance

Interpolation-based irrational model control design and stability analysis, Poussot-Vassal, Kergus, Vuillemin (2022).

Discussion

- These approaches consist in **approximating the ideal controller** $K^\star$

$$\frac{PK^\star}{1+PK^\star} = M$$

Data-Driven Control: Part Two of Two: Hot Take: Why not go with Models?, F. Dörfler, *IEEE Control Systems Magazine*, 2023

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- Choose wisely the reference model!
- All the presented methods (CbT, IFT, VRFT, L-DDC) are strongly linked to system identification

So why not go with models?

- Overcoming complexity (high dimension, non-linearities, uncertainty)

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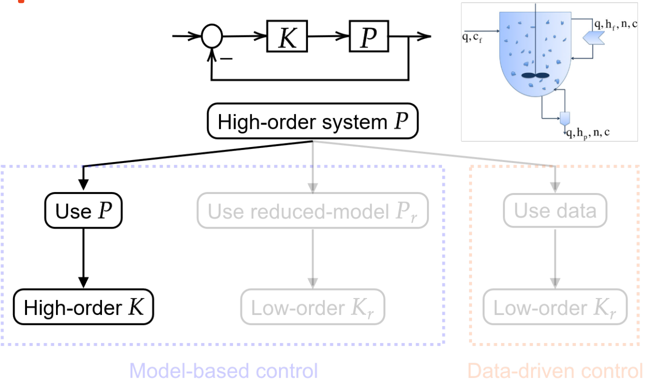
Application to infinite dimensional systems



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Laplace

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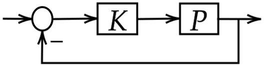

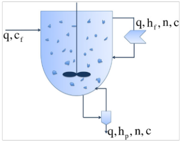

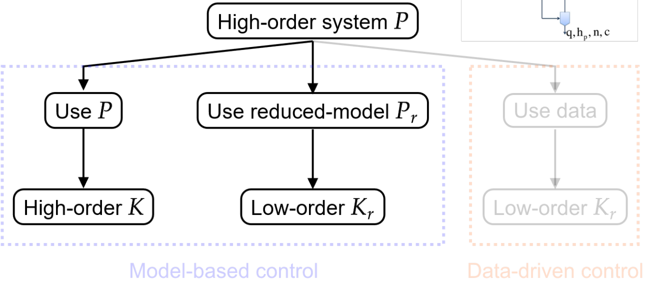

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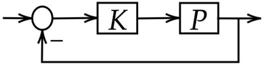

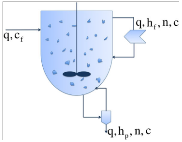

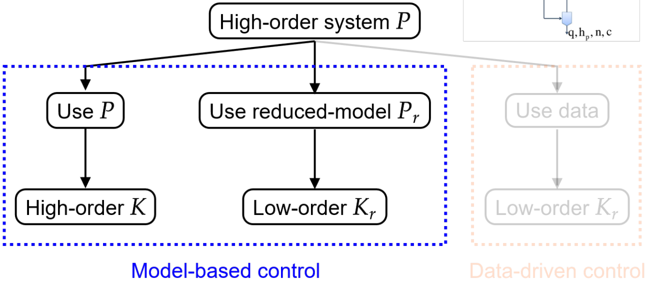

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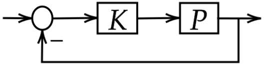

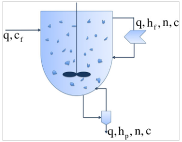

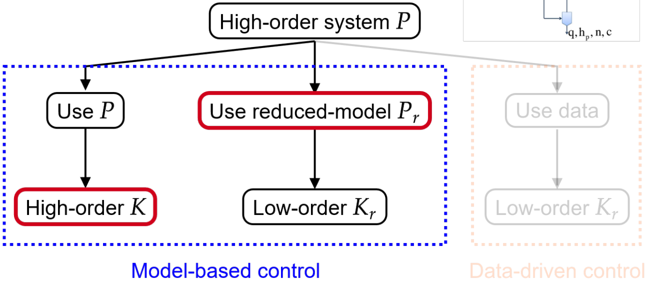

- Model Order Reduction is essential

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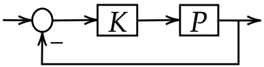

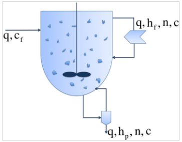

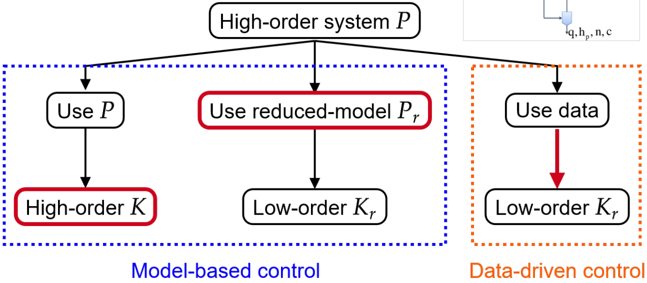

- Model Order Reduction is essential
- and it can also be used for data-driven control.

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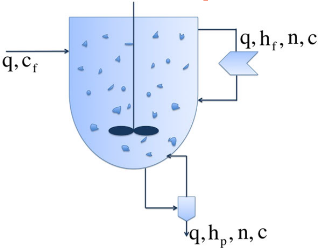
[3] *From reference model selection to controller validation: Application to Loewner Data-Driven Control*, Kergus, Olivi, Poussot-Vassal, Demourant, *IEEE Control Systems Letters*, 2019.

Case study: the continuous crystallizer



- **Goal:** stabilize the plant around $c_{ss} = 4.09 mol/L$
- Unstable system and sustained oscillations
- Linearization of the PDEs around c_{ss}

$$P(s) = \frac{\Delta c(s)}{\Delta c_f(s)} = \frac{p_{12}(s)}{p_{13}(s) + q_{12}(s)e^{-sk_f} + r_{12}(s)e^{-sk_p}}$$

A mathematical model for continuous crystallization, Rachah, Noll, Espitalier, Baillon, 2016.

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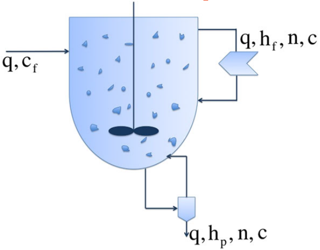
Case study: the continuous crystallizer



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Frequency-domain data easily accessible $\rightarrow$ **ideal case for L-DDC**

$N = 500$ frequencies, logspaced between 10^{-3} and 1 rad.s^{-1}

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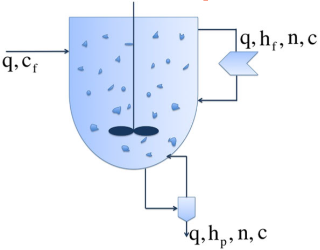

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$\{P(i\omega_i)\} \longrightarrow$ Build M $\longrightarrow$ Approximate $K^\star$ $\longrightarrow$ Reduction of K

- Build M: **performant AND reachable** (Instability estimation)
- Approximate $K^\star$: **Loewner interpolation**
- Reduction of K: **Under internal stability constraint**

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$$\begin{cases} \mathbf{y}_{z_i}^T \mathbf{P}(z_i) = 0 & \text{(RHP zeros)} \\ \mathbf{y}_{p_j} \mathbf{P}(p_j) = \infty & \text{(RHP poles)} \end{cases} \Rightarrow \boxed{\begin{cases} \mathbf{y}_{z_i}{}^T \mathbf{M}(z_i) = 0 \\ \mathbf{M}(p_j)\mathbf{y}_{p_j} = \mathbf{y}_{p_j} \end{cases}}$$

Achievable closed-loops satisfies interpolatory conditions at the RHP poles and zeros

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Achievable performance of multivariable systems with unstable zeros and poles, Havre, Skogestad, 2001.

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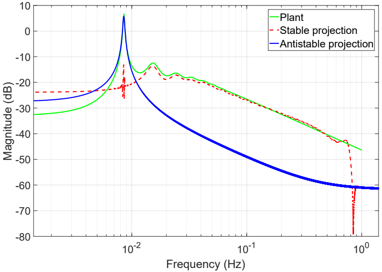

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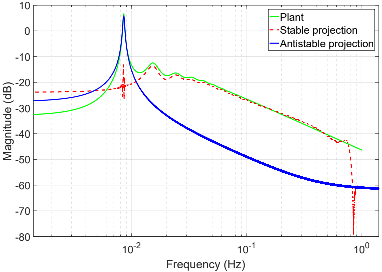
The system is unstable.



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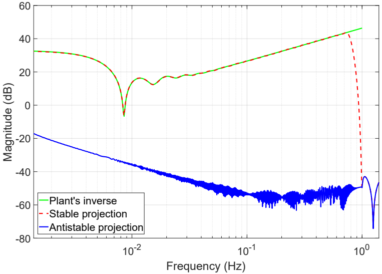
Achievable closed-loops satisfies interpolatory conditions at the RHP poles and zeros

Is the system minimum phase or not?

$$\mathbf{P}^{-1}(\imath\omega) = \mathbf{P}_s^{-1}(\imath\omega) + \mathbf{P}_{as}^{-1}(\imath\omega)$$

The system is minimum phase.



Achievable performance of multivariable systems with unstable zeros and poles, Havre, Skogestad, 2001.

Model-free closed-loop stability analysis: A linear functional approach, Cooman, Seyfert, Olivi, Chevillard, Baratchart, 2018

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Where are the system's instabilities?

Estimating unstable poles in simulations of microwave circuits, Cooman, Seyfert, Amari, 2018.

Laplace

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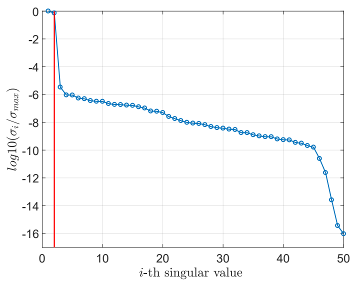
$$\begin{cases} \mathbf{y}_{z_i}^T \mathbf{P}(z_i) = 0 \\ \mathbf{y}_{p_j} \mathbf{P}(p_j) = \infty \end{cases} \quad \Rightarrow \quad \boxed{\begin{cases} \mathbf{y}_{z_i}{}^T \mathbf{M}(z_i) = 0 \\ \mathbf{M}(p_j)\mathbf{y}_{p_j} = \mathbf{y}_{p_j} \end{cases}}$$

Where are the system's instabilities?

$$\begin{array}{ccccc} \mathbf{P}(\imath\omega) & = & \mathbf{P}_s(\imath\omega) & + & \mathbf{P}_{as}(\imath\omega) \\ \mathcal{L}_2 & = & \mathcal{H}_2 & \oplus & \mathcal{H}_2^{\perp} \end{array}$$

Principal Hankel Component analysis on P_{as}



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Laplace

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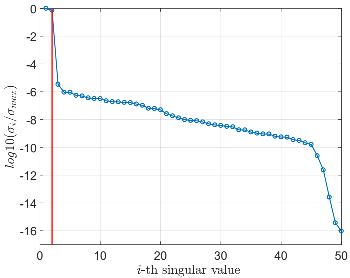
- Estimated RHP poles : $1.07 \times 10^{-4} \pm 0.852 \times 10^{-2}\jmath$
- RHP poles (direct search): $0.99 \times 10^{-4} \pm 0.89 \times 10^{-2}\jmath$

Principal Hankel Component analysis on P_{as}



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How to choose the reference model accordingly?

Step 1: Building a reference model

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How to choose the reference model accordingly?

$$M_{init}(s) = \frac{1}{1+\tau s}, \ \tau = 1s$$

$$\boxed{M = 1 - (1 - M_{init})B_p}$$

$$B_p(s) = \prod_{j=1}^{n_p} \frac{s - p_j}{s + p_j}$$

$$\forall j = 1 \dots n_p, \ B_p(p_j) = 0$$

$$\forall \omega, \ |B_p(\jmath\omega)| = 1$$

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Causality of the controller: The relative degree of the reference model cannot be smaller than the relative degree of the system.

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$\{P(i\omega_i)\}$ → Build M → Approximate $K^\star$ → Reduction of K

performant
AND reachable
(Instability estimation)

Loewner interpolation

Under internal
stability constraint

Laplace

Interlude: the Loewner framework

$$\text{Find } \mathbf{g} \text{ such that } \begin{cases} \mathbf{g}(\lambda_j) & = & \mathbf{w}_j, \; j = 1, \ldots, k \\ \mathbf{g}(\mu_i) & = & \mathbf{v}_i, \; i = 1, \ldots, q \end{cases}$$

Lagrangian form

$$\mathbf{g}(s) = \frac{\sum_{j=1}^{k_1} \frac{c_j \mathbf{w}_j}{s-\lambda_j}}{\sum_{j_1=1}^{k_1} \frac{c_j}{{}^{1}s-\lambda_j}}$$

A tutorial introduction to the Loewner framework for model reduction, Antoulas et al. (2017).

Interlude: the Loewner framework

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Loewner matrix

$$\mathbb{L} \in \mathbb{C}^{q \times k}$$

$$(\mathbb{L})_{i,j} = \frac{\mathbf{v}_i - \mathbf{w}_j}{\mu_i - \lambda_j}$$

Null space

$$\mathbb{L}\mathbf{c} = 0$$

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k_1} \end{bmatrix} \in \mathbb{C}^{k_1}$$

A tutorial introduction to the Loewner framework for model reduction, Antoulas et al. (2017).

Interlude: the Loewner framework

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y = \mathbf{C}\mathbf{x} \end{cases}$$

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$$

Objective: Find $\left(\hat{\mathbf{E}}, \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}\right)$ such that for a set of interpolation points $\{s_k\}$

$$\forall i,\ \hat{\mathbf{H}}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i$$

$$\forall j,\ \mathbf{l}_j\hat{\mathbf{H}}(\mu_j) = \mathbf{v}_j$$

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$$\boxed{\begin{cases} -\mathbb{L}\dot{\tilde{\mathbf{x}}} = -\mathbb{L}_s\tilde{\mathbf{x}} + \mathbf{V}\mathbf{u} \\ \tilde{\mathbf{y}} = \mathbf{W}\tilde{\mathbf{x}} \end{cases}}$$

Objective: Find $\left(\hat{\mathbf{E}}, \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}\right)$ such that for a set of interpolation points $\{s_k\}$

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- $\mathbb{L}$, $\mathbb{L}_s$, $\mathbf{V}$ and $\mathbf{W}$ are data matrices
- Model order reduction based on SVD of the pencil $(\mathbb{L}, \mathbb{L}_s)$

$$(\mathbb{L})_{i,j} = \frac{\mathbf{v}_i\mathbf{r}_i - \mathbf{l}_j\mathbf{w}_j}{\mu_i - \lambda_j} \quad (\mathbb{L}_s)_{i,j} = \frac{\mu_i\mathbf{v}_i\mathbf{r}_i - \lambda_j\mathbf{l}_j\mathbf{w}_j}{\mu_i - \lambda_j}$$

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- Factorization in terms of the tangential generalized controllability $\mathcal{R}$ and observability $\mathcal{O}$ matrices:

$$\mathbb{L} = -\mathcal{O}\mathbf{E}\mathcal{R} \quad \mathbb{L}_s = -\mathcal{O}\mathbf{A}\mathcal{R} \quad \mathbf{V} = \mathbf{C}\mathcal{R} \quad \mathbf{W} = \mathcal{O}\mathbf{B}$$

$$\mathbf{A}\mathcal{R} + \mathbf{B}\mathbf{R} = \mathbf{E}\mathcal{R}\mathbf{\Lambda} \qquad \mathcal{O}\mathbf{A} + \mathbf{L}\mathbf{C} = \mathbf{M}\mathcal{O}\mathbf{E}$$

Step 2: Controller identification and reduction

Objective: obtain a rational model $\mathbf{K} = (E, A, B, C, D)$ such that:

$$\forall i = 1 \dots N, \mathbf{K}(\imath\omega_i) = \mathbf{K}^\star(\imath\omega_i) = \frac{M(\jmath\omega_i)}{P(\jmath\omega_i)(1 - M(\jmath\omega_i))}.$$

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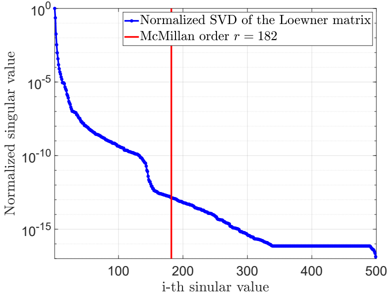


Figure: SVD of $\mathbb{L}$.

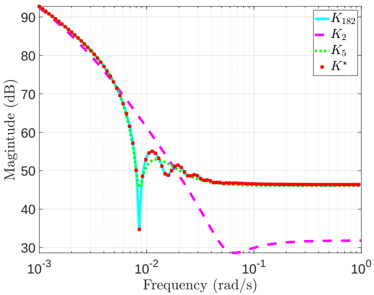


Figure: Obtained controllers.

Laplace

Step 3: Closed-loop stability analysis

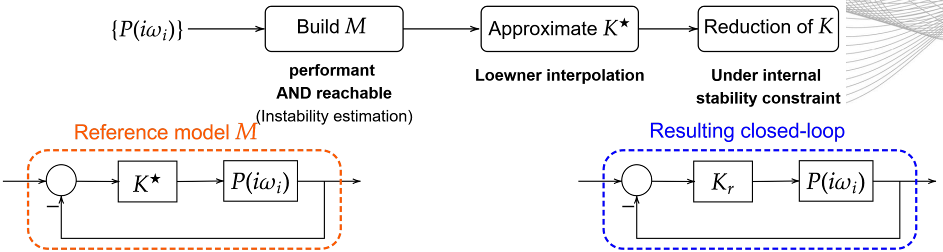
$\{P(i\omega_i)\} \longrightarrow$ Build M $\longrightarrow$ Approximate $K^\star$ $\longrightarrow$ Reduction of K

Build M: **performant AND reachable** (Instability estimation)

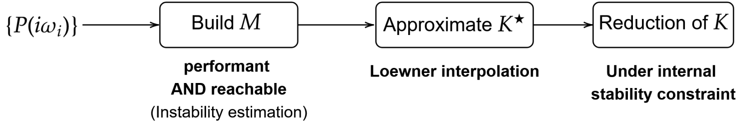
Approximate $K^\star$: **Loewner interpolation**

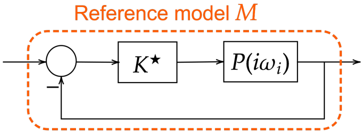
Reduction of K: **Under internal stability constraint**

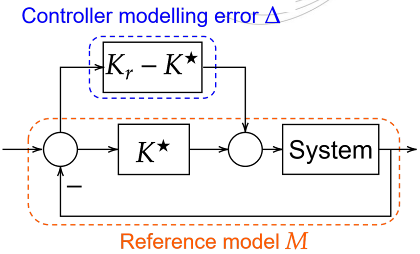
Step 3: Closed-loop stability analysis



Step 3: Closed-loop stability analysis

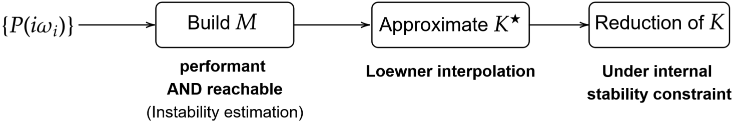


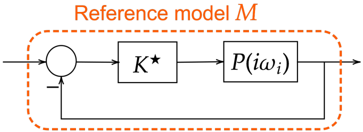


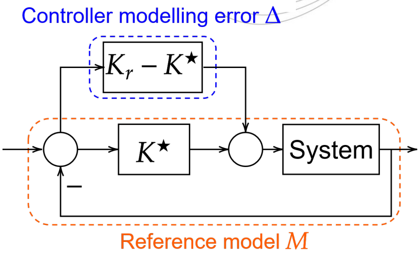


Laplace

Step 3: Closed-loop stability analysis





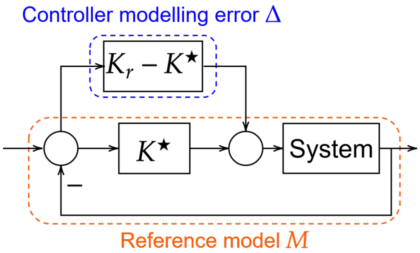


Application of the small-gain theorem

The closed-loop is well-posed and internally stable for all stable $\Delta = K - K^\star$ such that $\|\Delta\|_\infty \leq \beta$ if and only if $\|(1-M)P\|_\infty < \frac{1}{\beta}$

Data-driven controller validation, Van Heusden, Karimi, Bonvin, *IFAC Proceedings*, 2009.

Step 3: Closed-loop stability analysis

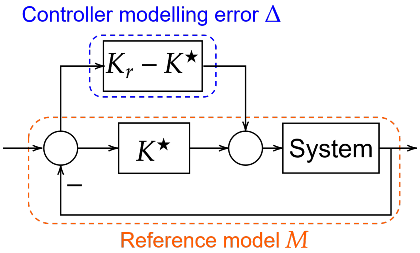


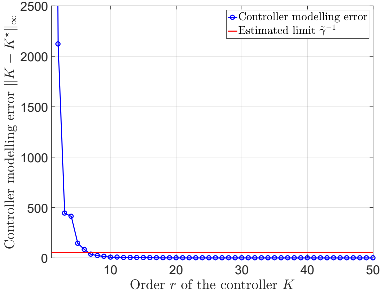
- Limiting the controller modelling error allows to ensure closed-loop internal stability!

$$\|K_r - K^\star\|_\infty \leq \beta$$

$$\frac{1}{\beta} \approx \max |(1 - M(\imath\omega_i))P(\imath\omega_i)|$$

Step 3: Closed-loop stability analysis





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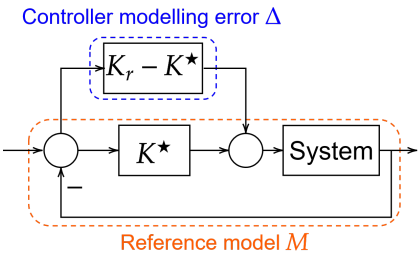
- We can reduce up to $r = 7$

Step 3: Closed-loop stability analysis



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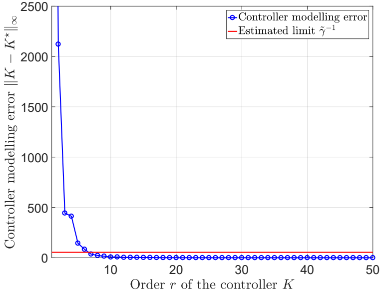
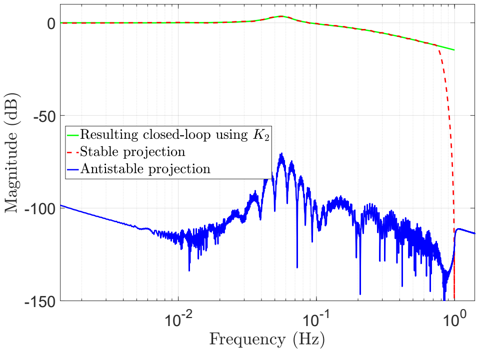


- We can reduce up to $r = 7$
- **Conservative criteria!**

Alternative closed-loop stability analysis

$$H(\jmath\omega_i) = \frac{P(\jmath\omega_i)K_r(\jmath\omega_i)}{1 + P(\jmath\omega_i)K_r(\jmath\omega_i)}$$

1st option



Model-free closed-loop stability analysis: A linear functional approach, Cooman et al., 2018

Laplace

Alternative closed-loop stability analysis

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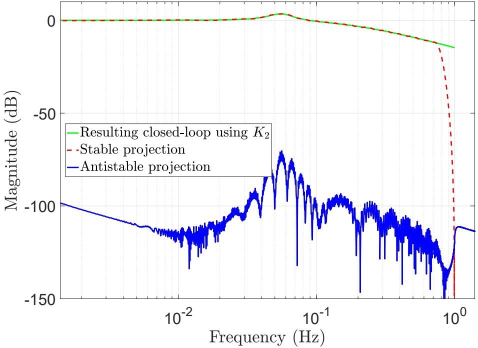

2nd option

1. Loewner interpolation: $\hat{H}(\jmath\omega_i) = H(\jmath\omega_i)$

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Interpolation-based infinite dimensional model control design and stability analysis, Poussot, Kergus, Vuillemin, 2020.

Laplace

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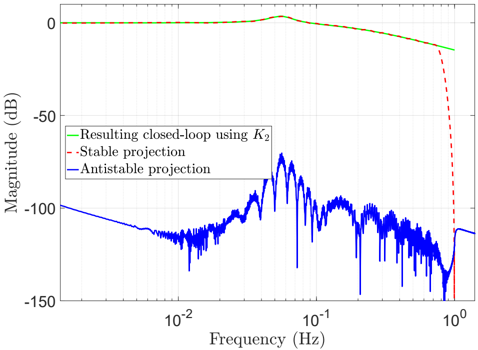

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1. Loewner interpolation: $\hat{H}(\jmath\omega_i) = H(\jmath\omega_i)$
2. Stable projection on $\mathcal{RH}_\infty$:
$$\hat{H}_s = \underset{H \in \mathbb{S}^+_{n,n_i,n_o}}{\arg\min} \|H - \hat{H}\|_\infty$$

Model-free closed-loop stability analysis: A linear functional approach, Cooman et al., 2018
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On the closest stable descriptor system in the respective spaces $\mathcal{RH}_2$ and $\mathcal{RH}_\infty$, Köhler, 2014.

Laplace

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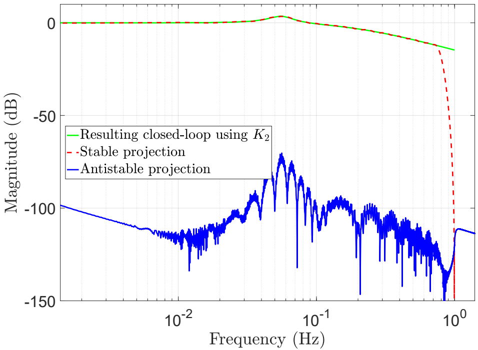

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   $$\hat{H}_s = \underset{H \in \mathbb{S}^+_{n,n_i,n_o}}{\text{arg min}} \|H - \hat{H}\|_\infty$$
3. Stability index $S = \|\hat{H}_s - \hat{H}\|_\infty$

$$\boxed{S = 4.3511 \cdot 10^{-6}}$$

Model-free closed-loop stability analysis: A linear functional approach, Cooman et al., 2018
Interpolation-based infinite dimensional model control design and stability analysis, Poussot, Kergus, Vuillemin, 2020.
On the closest stable descriptor system in the respective spaces $\mathcal{RH}_2$ and $\mathcal{RH}_\infty$, Köhler, 2014.

Laplace

Results

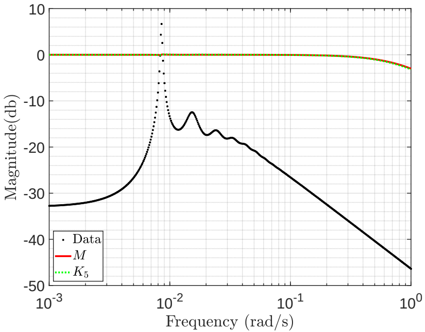


Figure: Closed-loop transfer functions.

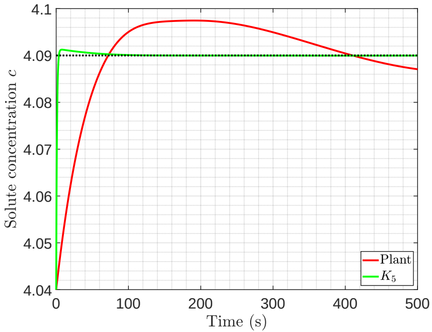
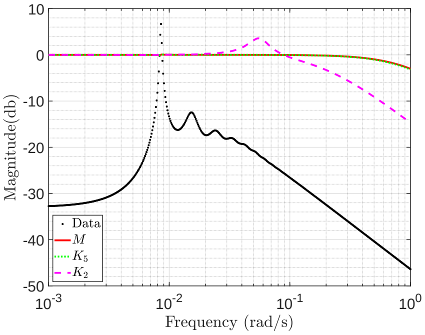


Figure: Time-domain simulation.

Results



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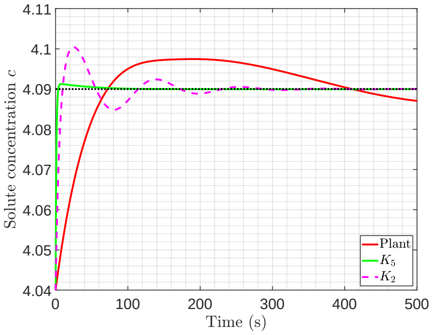


Figure: Time-domain simulation.

$\Rightarrow$ Impact of the **complexity-accuracy trade-off**

Results

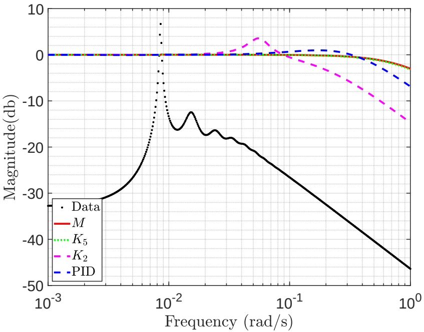
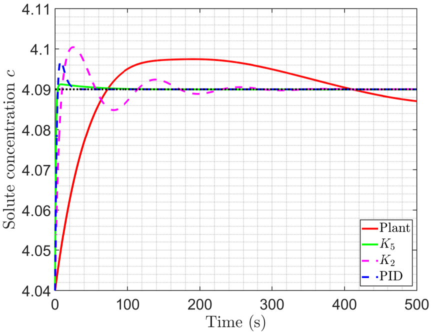


Figure: Closed-loop transfer functions.



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→ Impact of the reference model (comparison with a robust PID)

Structured H control of infinite-dimensional systems, Apkarian and Noll, 2018.

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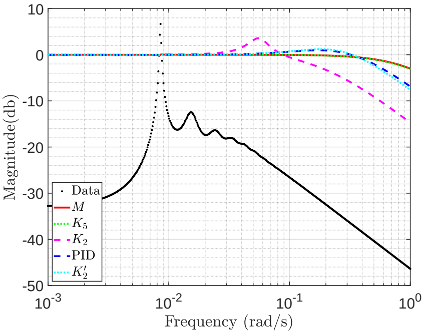

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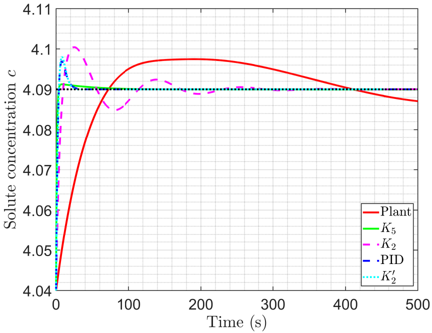

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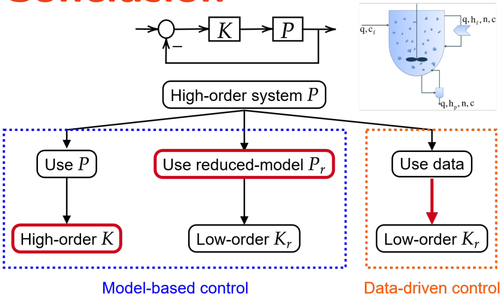

The Loewner framework can be used as a central tool for the control of infinite dimensional transfer functions.

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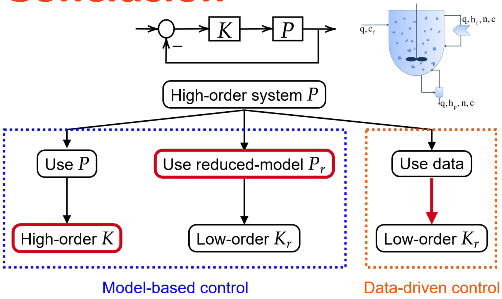

The Loewner framework can be used as a central tool for the control of infinite dimensional transfer functions.

	Model-based design	L-DDC
Method	more steps	direct
Controller structure	fixed order/poles	linear
Specifications	flexible (robust)	not flexible (only stability)
Stability guarantees	for P_r	conservative or not embedded

Discussions

For which kind of systems?

- Complex systems (high dimension)

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Laplace

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Relation with other data-driven techniques:

- **Model-Free Control:**
  - $y = y^{\star}$, i.e. $M = I$
  - Online estimation of an ultra-local model $\rightarrow$ adaptation of the ideal controller ($K = K^{\star}$ locally)

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  - Online estimation of an ultra-local model $\rightarrow$ adaptation of the ideal controller ($K = K^\star$ locally)
- **Willem's lemma based approaches:** controller identification based on the data-based representation, but different specifications format

Laplace

Extending L-DDC through multivariate interpolation

- The Loewner framework is a model order reduction approach
  Full-Order Model (FOM) $\rightarrow$ Reduced-Order Model (ROM)

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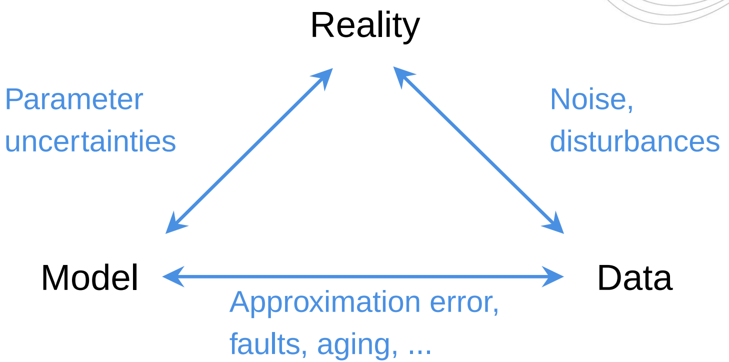

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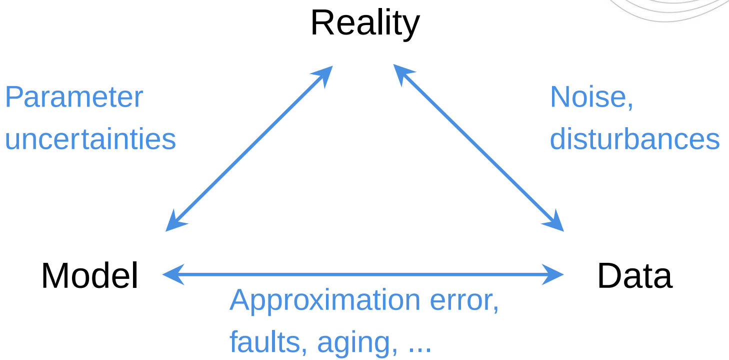

- **Multivariate modeling** is very helpful!

The Loewner framework in the 1-D case

Find $\mathbf{g}$ such that $\begin{cases} \mathbf{g}(\lambda_j) & = & \mathbf{w}_j, \ j = 1, \ldots, k \\ \mathbf{g}(\mu_i) & = & \mathbf{v}_i, \ i = 1, \ldots, q \end{cases}$

Lagrangian form

$$\mathbf{g}(s) = \frac{\sum_{j=1}^{k_1} \frac{c_j \mathbf{w}_j}{s-\lambda_j}}{\sum_{j_1=1}^{k_1} \frac{c_j}{{}^1 s-\lambda_j}}$$

A tutorial introduction to the Loewner framework for model reduction, Antoulas et al. (2017).

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Loewner matrix

$$\mathbb{L} \in \mathbb{C}^{q \times k}$$

$$(\mathbb{L})_{i,j} = \frac{\mathbf{v}_i - \mathbf{w}_j}{\mu_i - \lambda_j}$$

Null space

$$\mathbb{L}\mathbf{c} = 0$$

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k_1} \end{bmatrix} \in \mathbb{C}^{k_1}$$

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Extension to the 2D case

Find g such that $\begin{cases} \mathbf{g}({}^1\lambda_{j_1}, {}^2\lambda_{j_2}) & = & \mathbf{w}_{j_1,j_2}, \; j_1 = 1, \ldots, k_1, \; j_2 = 1, \ldots, k_2 \\ \mathbf{g}({}^1\mu_{i_1}, {}^2\mu_{i_2}) & = & \mathbf{v}_{i_1,i_2}, \; i_1 = 1, \ldots, q_1, \; i_2 = 1, \ldots, q_2 \end{cases}$

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The general n-D case

$$
\begin{array}{rcl}
\mathbb{C}^{k_1} \times \mathbb{C}^{q_1} \times \ldots \times \mathbb{C}^{k_n} \times \mathbb{C}^{q_n} \times \mathbb{C}^{(k_1+q_1)\times\cdots\times(k_n+q_n)} & \longrightarrow & \mathbb{C}^{Q\times K} \\
\left({}^{1}\lambda_{j_1}, {}^{1}\mu_{i_1}, \ldots, {}^{n}\lambda_{j_n}, {}^{n}\mu_{i_n}, \textbf{tab}_{nD}\right) & \longmapsto & \mathbb{L}_{nD}
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The curse of dimensionality

$$\mathbb{L}_{nD} \in \mathbb{C}^{Q\times K}$$

$Q = q_1 q_2 \ldots q_n$ and $K = k_1 k_2 \ldots k_n$

Laplace

Taming the curse of dimensionality

- Iterative solution based on variable decoupling

$$\mathbf{g}({}^1s, {}^2s) = \frac{\sum_{j_1,j_2=1}^{k_1,k_2} \frac{c_{j_1,j_2}\mathbf{w}_{j_1,j_2}}{({}^1s - {}^1\lambda_{j_1})({}^2s - {}^2\lambda_{j_2})}}{\sum_{j_1,j_2}^{k_1,k_2} \frac{c_{j_1,j_2}}{({}^1s - {}^1\lambda_{j_1})({}^2s - {}^2\lambda_{j_2})}}$$

$$\mathbf{g}({}^1s, {}^2\lambda_k) = \frac{\mathsf{num}_{{}^2\lambda_k}({}^1s)}{\pi_{{}^2\lambda_k}({}^1s)\sum_{j_1=1}^{k_1} \frac{c_{j_1,k}}{({}^1s - {}^1\lambda_{j_1})}}$$

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Decoupling process

$$\mathcal{N}(\mathbb{L}_{2D}) = \mathbf{c}_{2D}^{\top} = \left[\begin{array}{ccc|c|ccc} c_{1,1} & \dots & c_{1,k_2} & \dots & c_{k_1,1} & \dots & c_{k_1,k_2} \end{array}\right]^T$$

$$\textbf{tab}_{2D} = \left[\begin{array}{c|cccc} & {}^2s_1 & {}^2s_2 & \dots & {}^2s_n \\ \hline {}^1s_1 & h_{1,1} & h_{1,2} & \dots & h_{1,m} \\ {}^1s_2 & h_{2,1} & h_{2,2} & \dots & h_{2,m} \\ \vdots & \vdots & \vdots & & \vdots \\ {}^1s_n & h_{n,1} & h_{n,2} & \dots & h_{n,m} \end{array}\right] \quad \left[\begin{array}{c|cccc} & {}^2\lambda_1 & {}^2\lambda_2 & \dots & {}^2\lambda_{k_2} \\ \hline {}^1\lambda_1 & c_{1,1} & c_{1,2} & \dots & c_{1,k_2} \\ {}^1\lambda_2 & c_{2,1} & c_{2,2} & \dots & c_{2,k_2} \\ \vdots & \vdots & \vdots & & \vdots \\ {}^1\lambda_{k_1} & c_{k_1,1} & c_{k_1,2} & \dots & c_{k_1,k_2} \end{array}\right]$$

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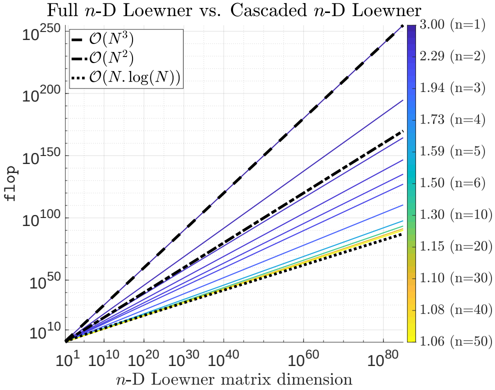
- **Without decoupling:** 1 $2D$ problem with $\mathbb{L}_{nD} \in \mathbb{C}^{q_1 q_2 \times k_1 k_2}$
- **With decoupling:** k_1 $1D$ problems with $\mathbb{L}_{1D} \in \mathbb{C}^{q_2 \times k_2}$ and 1 with $\mathbb{L}_{1D} \in \mathbb{C}^{q_1 \times k_1}$

On the Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality, Antoulas, Gosea, Poussot-Vassal (2024).

Taming the curse of dimensionality

log-log scale



Computational issue

$Q \times K$ matrix SVD `flop` is

- QK^2 (if $Q > K$)
- N^3 (if $Q = K = N$)

Storage issue

$Q \times K$ matrix storage is

- in real double
- in complex double

On the Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality, Antoulas, Gosea, Poussot-Vassal (2024).

Laplace

Connection to the Kolmogorov Superposition Theorem

Kolmogorov, Arnol'd, Kahane, Lorentz, and Sprecher

For any $n \in \mathbb{N}$, $n \geq 2$, there exist real numbers $\lambda_1, \ldots, \lambda_n$ and continuous functions $\Phi_k : \mathbb{I} \mapsto \mathbb{R}$, $k = 1, \ldots, 2n+1$, with the property that for every continuous function $f : \mathbb{I}^n \mapsto \mathbb{R}$, there exists a continuous function $g : \mathbb{R} \mapsto \mathbb{R}$ such that:

$$\forall (x_1, \ldots, x_n) \in \mathbb{I}^n, \quad f(x_1, \ldots, x_n) = \sum_{k=1}^{2n+1} g(\lambda_1 \Phi_k(x_1) + \cdots + \lambda_n \Phi_k(x_n))$$

Hilbert 13: Are there any genuine continuous multivariate real-valued functions?, S. Morris, (2021).

KST-like representation for rational functions

- Thanks to the decoupling process, one can write :

$$\mathbf{H}(s,t)=\frac{\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\frac{c_{j_1,j_2}\mathbf{w}_{j_1,j_2}}{\left(s-{}^1\lambda_{j_1}\right)\left(t-{}^2\lambda_{j_2}\right)}}{\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\frac{c_{j_1,j_2}}{\left(s-{}^1\lambda_{j_1}\right)\left(t-{}^2\lambda_{j_2}\right)}}$$

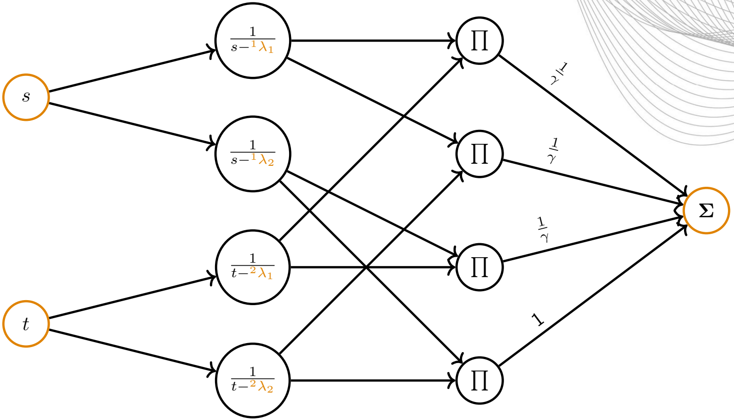
$$=\frac{\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\exp\left(\log \mathbf{w}_{j_1,j_2}+\log\frac{\textbf{Bary}^s_{j_1}}{\left(s-{}^1\lambda_{j_1}\right)}+\log\frac{\textbf{Bary}^t_{j_2}}{\left(t-{}^2\lambda_{j_2}\right)}\right)}{\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\exp\left(\log\frac{\textbf{Bary}^s_{j_1}}{\left(s-{}^1\lambda_{j_1}\right)}+\log\frac{\textbf{Bary}^t_{j_2}}{\left(t-{}^2\lambda_{j_2}\right)}\right)}$$

- Composition and superposition of one-variable functions

On the Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality, Antoulas, Gosea, Poussot-Vassal (2024).

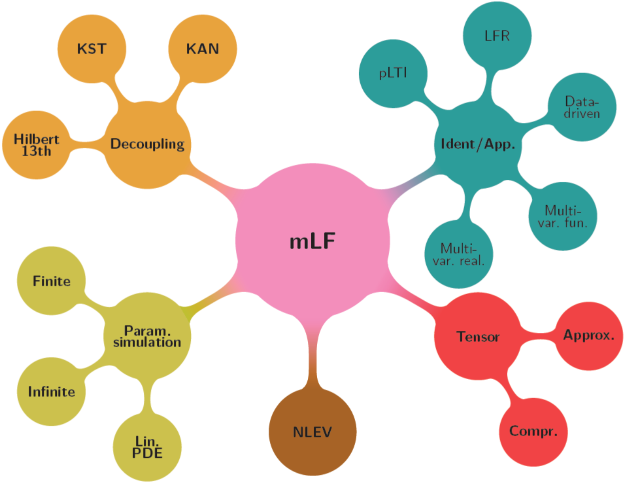
Laplace

Connection with neural networks



NN with barycentric activation functions and product/sum

KAN: Kolmogorov-Arnold Networks, Liu et al. (2024).



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